Yexin Qu on enriched category theory

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1. Kan extension
2. End + coend
3.

1. Define left Kan extension.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] \arrow[r, phantom, shift right=4, "\eta\Downarrow"] & \mathcal{E} \arrow[r, "G"] & \mathit{Set} \\
\mathcal{D} \arrow[ur, dashed, "\mathrm{Lan}_K F"] \arrow[urr, dashed, "\mathrm{Lan}_K GF"'] & &
\end{tikzcd}$$

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K(-),-) \cong \mathcal{E}^{\mathcal{C}}(-,K^*(-))$$

A functor $\mathcal{C} \xrightarrow{F} \mathit{Set}$ is <u>co-representable</u> if

$F \cong h^c : \mathcal{C} \longrightarrow \mathit{Set}$ for some $c \in \mathrm{Ob}\,\mathcal{C}$

$\qquad\qquad x \longmapsto \mathcal{C}(c,x)$

A functor $F : \mathcal{C} \longrightarrow \mathit{Set}^{op}$ is rep if ...

A left Kan extension is <u>pointwise</u> if it preserved by all corep functors from $\mathcal{E}$, i.e. if

$\mathrm{Lan}_K GF = G\,\mathrm{Lan}_K F$ for all corep functors G.

2. Ends + coends $\quad H : J^{op} \times J \longrightarrow \mathcal{C}$ $\quad$ J small, $\mathcal{C}$ complete

For any $j \xrightarrow{f} j'$ in J we have

$$\begin{tikzcd}
\text{pull-back} \arrow[r] \arrow[d] & H(j,j) \arrow[d, "f_*"] \\
H(j',j') \arrow[r, "f^*"] & H(j,j')
\end{tikzcd}$$

$$H(j,j) \xrightarrow{\varphi_*} \prod_{f : j \to j'} H(j, f(j))$$

$$\begin{tikzcd}
\prod_{j \in J} H(j,j) \arrow[r, "\varphi_*"] & \prod H(j, f(j)) \\
\prod_{j \in J} H(j',j') \arrow[ur, "\varphi^*"'] &
\end{tikzcd}$$

Def equalizer $\longrightarrow \prod H(j,j) \overset{\varphi_*}{\underset{\varphi_*}{\rightrightarrows}} \prod H(j,j')$

$\| $

$\int_J H(j,j) =$ end

Dually for $\mathcal{C}$ cocomplete, J small, we have

$\coprod_f H(\mathrm{dom} f, \mathrm{cod} f) \overset{\varphi^*}{\underset{\varphi_*}{\rightrightarrows}} \coprod H(j,j) \longrightarrow$ coequalizer

coend $= \int^J H(j,j)$

Formula for left Kan extension for $\mathcal{C}$ small and $\mathcal{E}$ co-complete

set

$$\mathrm{Lan}_K F(d) = \int^J \mathcal{D}(K(c),d) \cdot F(c)$$

for $d \in \mathcal{D}$

3. Monoidal categories

Def $\mathcal{C}$ is monoidal if it has

1) Binary op $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$
2) Unit object 1 with natural isos

$a_{x,y,z} \quad (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$

$\lambda: \ 1 \otimes x \longrightarrow x$

$\rho: \ x \otimes 1 \longrightarrow x$

$(\mathcal{C}, \otimes, 1)$ is <u>symmetric</u> if $\exists$ nat iso

$$x \otimes y \xrightarrow{T_{x,y}} y \otimes x$$

It is closed if $- \otimes x$ has right adjoint, the internal hom $\underline{\mathcal{C}}(x,-)$

Ex $\mathcal{C} = (\mathrm{Vect}_k, \oplus, 0)$ with embedding as morphisms

$$\mathcal{C}(0, \mathcal{C}(x,y)) \overset{?}{=} \mathcal{C}(x,y) \quad \mathcal{C} \text{ not closed}$$

$\mathcal{C}(0, \mathcal{C}(x,y))$ is $\cong 0$; $\mathcal{C}(x,y) \not\cong 0$

4. Enriched category

<u>Def</u> Let $\mathcal{V} = (\mathcal{V}_0, \otimes, 1)$ be a monoidal cat

a <u>$\mathcal{V}$-category</u> (or category enriched over $\mathcal{V}$)

1) $\mathcal{C}$ has a collection of objects $ob(\mathcal{C})$ and

2) for any $X, Y \in \mathcal{C}$, we have a morphism object $\mathcal{C}(X,Y)$ in $\mathcal{V}_0$ with $1 \to \mathcal{C}(X,X)$ a morphism in $\mathcal{V}_0$ for each X.

3) For $X,Y,Z \in \mathcal{C}$ we have composition

$$\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

with suitable properties

We can also define enriched functors + natural transformations.

5. An enriched functor $F : \mathcal{D} \to \mathcal{C}$ consists of $F : \mathrm{ob}\,\mathcal{D} \to \mathrm{ob}\,\mathcal{C}$

$$\mathcal{D}(X,Y) \longrightarrow \mathcal{C}(F(X),F(Y))$$

A <u>V-natural transformation</u> $T : F \to G$ assigns to each $X \in \mathcal{D}$ a morphism $T_X : 1 \longrightarrow \mathcal{C}(F(X),G(X))$ and

$$\begin{tikzcd}
\mathcal{D}(X,Y) \arrow[r, "T_Y \otimes F"] \arrow[d, "G \otimes T_X"'] & \mathcal{C}(F(Y),G(Y)) \otimes \mathcal{C}(F(X),F(Y)) \arrow[d] \\
\mathcal{C}(G(X),G(Y)) \otimes \mathcal{C}(F(X),G(X)) \arrow[r] & \mathcal{C}(F(X),G(Y))
\end{tikzcd}$$

6. <u>Day convolution</u> Let $\mathcal{D} = (\mathcal{D}_0, \oplus, 0)$ be a small SMC enriched over $\mathcal{V} = (\mathcal{V}_0, \otimes, 1)$, a cocomplete CSMC

Let $X, Y \in [\mathcal{D}, \mathcal{V}]$ enriched functors

Let $X \square Y$ be the left Kan extension

$$\begin{tikzcd}
\mathcal{D} \times \mathcal{D} \arrow[r, "X \times Y"] \arrow[d, "\oplus"'] & \mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \\
\mathcal{D} \arrow[urr, dashed, "X \square Y = \mathrm{Lan}_{\oplus}(X \otimes Y)"'] & &
\end{tikzcd}$$

Thus $([\mathcal{D},\mathcal{V}], \square, \mathbb{I})$ is CSMC

where $\mathbb{I}: \mathcal{D} \longrightarrow \mathcal{V}$

$$D \longmapsto \mathcal{V}(0,D)$$

Example $\mathcal{D} = \mathcal{J}_G$ = Mandell-May category of orth reps of G

$\mathcal{V} = \mathcal{T}_G$ = pointed G-spaces

$$\begin{tikzcd}
\mathcal{J}_G \times \mathcal{J}_G \arrow[r, "X \times Y"] \arrow[d, "\oplus"'] & \mathcal{T}_G \times \mathcal{T}_G \arrow[r, "\wedge"] & \mathcal{T}_G \\
\mathcal{J}_G \arrow[urr, "X \wedge Y"'] & &
\end{tikzcd}$$

$$X \wedge Y = \mathrm{Lan}_{\oplus}(\wedge \circ X \times Y)$$